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Electromagnetic wave-fronts in anisotropic media as discontinuity surfaces. A new mechanical analogy: vortex sheets

Frentes de onda electromagnéticos como superficies de discontinuidad en medios anisótropos. Una nueva analogía mecánica: hojas de vórtices

C. Bellver-Cebreros[1,S*], M. Rodríguez-Danta[1]

1. Departamento de Física Aplicada III, Escuela Técnica Superior de Ingeniería, Universidad de Sevilla, Camino de los Descubrimientos, s/n, 41092-Seville, Spain.

(*) E-mail: consuelo@us.es S: miembro de SEDOPTICA / SEDOPTICA member

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ABSTRACT:

Under the hypothesis of electromagnetic waves propagation in inhomogeneous linear media with dielectric and magnetic anisotropy (in the absence of free charges and currents), analogies are presented between several discontinuity surfaces and wave-front propagation in these media. In particular, the "vortex sheet" model, borrowed from Fluid Mechanics is discussed.

Key words: Dielectric anisotropy, magnetic anisotropy, wave-fronts, eikonal, thin plate, vortex sheet, optical-mechanical analogies, electromagnetic couple density.

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1. Introduction

In a homogeneous and isotropic medium, without free charges and currents $(\rho_f = 0\,;\, \boldsymbol{j}_f = 0)$ the electromagnetic field is rigorously derivable from the theory of complex potential $V(\boldsymbol{r},t)$ [1]. The $\boldsymbol{E}$ and $\boldsymbol{H}$ fields contained in the complex plane, obtained from the Argand diagram [2] are perpendicular to the direction of propagation of the front.

This fruitful "scalar representation of electromagnetic fields" was initially used to derive expressions for momentum and energy densities in terms of the complex potential *V*, as an alternative to the vector potential $\boldsymbol{A}$. This representation is still in use today in the study of vortex singularities and, in general, in Singular and Structured Optics [3-7].

A desirable extension of the scalar theory to anisotropic media (with both dielectric and magnetic anisotropy) does not seem trivial, since in this case, unlike the previous one, the four fields involved $\boldsymbol{E}$, $\boldsymbol{D}$, $\boldsymbol{H}$ and $\boldsymbol{B}$ are not coplanar and consequently, ray and wave-front propagate in non-coincident directions.

In an attempt to maintain a field structure analogous to scalar theory, the representation of the fields in polar form, for inhomogeneous media with dielectric and magnetic anisotropy, was proposed [8]:

$$\boldsymbol{E} = \boldsymbol{E}_0(\boldsymbol{r},t)\, e^{iS}\;; \quad \boldsymbol{B} = \boldsymbol{B}_0(\boldsymbol{r},t)\, e^{iS}\;; \tag{1}$$

$$\boldsymbol{D} = \boldsymbol{D}_0(\boldsymbol{r},t)\, e^{iS}\;; \quad \boldsymbol{H} = \boldsymbol{H}_0(\boldsymbol{r},t)\, e^{iS} \tag{2}$$

where the amplitudes $\boldsymbol{E}_0, \boldsymbol{D}_0, \boldsymbol{H}_0$ and $\boldsymbol{B}_0$ are given as any vector function of position and time and *S* is the eikonal or phase, which is also any real function of position and time. Substituting these fields into Maxwell's equations, with $\rho_f = 0\,;\, \boldsymbol{j}_f = 0$, and separating real and imaginary parts, in addition to Maxwell´s equations for the amplitudes, the equations for the phase evolution (wave-front) are obtained:

$$\nabla S \wedge \boldsymbol{E}_0 + \frac{\partial S}{\partial t}\boldsymbol{B}_0 = 0;\;\; \nabla S \wedge \boldsymbol{H}_0 - \frac{\partial S}{\partial t}\boldsymbol{D}_0 = 0 \tag{3}$$

$$\nabla S \cdot \boldsymbol{B}_0 = 0;\;\; \nabla S \cdot \boldsymbol{D}_0 = 0 \tag{4}$$

which, although formally identical to the equations of propagation of locally plane waves in isotropic media, the amplitudes $\boldsymbol{E}_0, \boldsymbol{B}_0, \boldsymbol{D}_0, \boldsymbol{H}_0$ are not coplanar.

Having ruled out the possibility of deriving the field from a single solenoidal vector potential, and thus of extending the powerful theory of complex variable functions to wave analysis and propagation in anisotropic media, any qualitative study of the mechanical behavior of the front (in its ability to transmit momentum, angular momentum and energy densities) requires the use of analogies borrowed from other branches of physics.

In an earlier paper [9], under the same hypothesis of anisotropic media without free charges and currents, and applying the balance equations of continuum mechanics, the asymmetry of the Maxwell stress tensor $(\overline{\overline{T}} = \overline{\overline{T}}^S + \overline{\overline{T}}^A$ or $T_{jk} = T^S_{(jk)} + T^A_{[jk]})$ was identified with electromagnetic couple density ($\boldsymbol{C}$). Its expression in vector form:

$$\boldsymbol{C} = \boldsymbol{D} \wedge \boldsymbol{E} + \boldsymbol{B} \wedge \boldsymbol{H} = \frac{d\boldsymbol{M}_0}{d\tau} \tag{5}$$

coincides with that proposed by Durand and Landau [10-11], and clearly demonstrates its biunivocal relationship with the anisotropy of the medium: non-zero $\boldsymbol{C}$ implies an anisotropic material.

As a continuation of the ideas set out in [9], and considering a point P on the front, moving with the speed of energy propagation (ray velocity $\mathbf{v} = (\boldsymbol{E} \times \boldsymbol{H})/W$, where W is the electromagnetic energy density), and from equations (3) and (4), it is easy to prove that

$$\frac{\partial S}{\partial t}\boldsymbol{C} = (\nabla S \wedge \mathbf{v})W \tag{6}$$

which demonstrates the orthogonality of the $\boldsymbol{C}$ vector to the plane formed by ∇S (normal to the wave-front) and the ray (see Fig. 1).

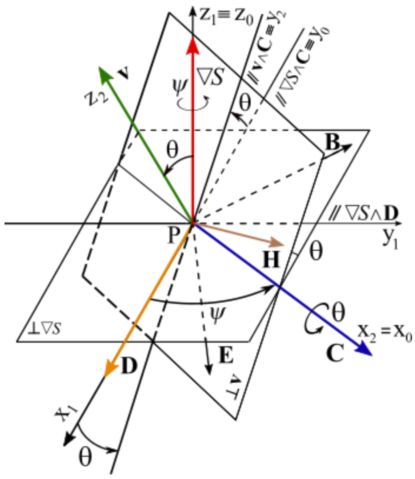

Fig. 1. Electromagnetic fields and $\boldsymbol{C}$ vector in media with dielectric and magnetic anisotropy.

On the other hand, defining (following Luneburg [12]) the wave-front as "any surface in space (x, y, z) over which, for an instant t, the electromagnetic field is discontinuous", its time evolution is governed by the application of the laws of continuum physics to the interfaces of electromagnetic fields. Models of discontinuity surfaces appear in many other fields of science; various analogies suitable in each case to the constitutive equations of the medium under study have been proposed. In [9] for linear anisotropic media with constitutive equations, $\boldsymbol{D} = \varepsilon_0 \vec{\vec{\varepsilon}} \boldsymbol{E}$; $\boldsymbol{B} = \mu_0 \vec{\vec{\mu}} \boldsymbol{H}$, the wave-front model assimilated to a mechanically stiff *"thin plate"* or *"double layer"*, responsible for bending and torsion and transmitting the torque related to **C,** seems appropriate.

2. Discontinuity Surfaces

2.a. "Wave-front" discontinuity

The mechanical balance equations developed in the above-mentioned paper [9], are applied to a finite domain with boundary $\partial\tau$, for the particular case in which $\partial\tau \equiv S(\boldsymbol{r}, t) = 0$ is the eikonal (interface that separates the regions disturbed by the fields from the undisturbed ones). These equations (*equivalent to those of impulsive dynamics*) are written as follows:

2.a.1 Energy:

As an application of the Poynting theorem in local form, one has

$$\|\boldsymbol{E} \wedge \boldsymbol{H}\| \cdot \nabla S + \frac{\partial S}{\partial t}\|W\| = 0 \tag{7}$$

and from the definition of eikonal:

$$\frac{dS}{dt} = \frac{\partial S}{\partial t} + \mathbf{v} \cdot \nabla S = 0 \tag{8}$$

it is concluded (as mentioned above) that the energy propagates at velocity $\mathbf{v} = (\boldsymbol{E}\times\boldsymbol{H})/W$. It must be noted that $\|A\| = A^{+} - A^{-}$ denotes the jump that experiments any physical quantity A in traversing the wave-front in direction of its normal $\boldsymbol{n} = \nabla S / |\nabla S|$.

2.a.2 Linear momentum

The application of the theorem of linear momentum gives:

$$\nabla S \cdot \left\|\overline{\overline{T}}^{s}\right\| + \frac{1}{2}\|\boldsymbol{C}\|\times\nabla S - \|\boldsymbol{D}\times\boldsymbol{B}\|\frac{\partial S}{\partial t} = 0 \tag{9}$$

and taking into account that $\boldsymbol{n}\cdot\left\|\overline{\overline{T}}^{s}\right\| = \frac{1}{2}\boldsymbol{n}\times\|\boldsymbol{C}\| - W\,\boldsymbol{n}$ and $\boldsymbol{n}\cdot\left\|\overline{\overline{T}}^{A}\right\| = -\frac{1}{2}\,\boldsymbol{n}\times\|\boldsymbol{C}\|$, Eq. (9) reduces to

$$\nabla S\,\|W\| + \|\boldsymbol{D}\times\boldsymbol{B}\|\,\frac{\partial S}{\partial t} = 0 \tag{10}$$

i.e. the front propagates under the action of a net force per unit volume ($-\|W\|\,\nabla S$) in the direction of its normal and a torque perpendicular to $\boldsymbol{n}$.

2.a.3 Angular momentum

The application of the angular momentum theorem, as already mentioned, leads to identify the volumetric momentum density $\boldsymbol{C}$ with the antisymmetric part of the Maxwell stress tensor:

$$C_i = \varepsilon_{ijk}\, T_{[jk]} \equiv (\boldsymbol{D}\wedge\boldsymbol{E} + \boldsymbol{B}\wedge\boldsymbol{H})_i \tag{11}$$

This term vanishes only when the medium is isotropic; its evolution is governed by Eq. (6).

2.b. Double dielectric layer analogy

The interface between two dielectric media and thus the wave-front, can be considered as a dielectric double layer (See Fig. 2 a). Indeed: if we have a material medium of polarization vector $\boldsymbol{P} = d\boldsymbol{p}/d\tau$ (dipole moment per unit volume), the scalar potential at any point M is given by:

$$V(\boldsymbol{r}) = k_e \int_{\tau} \frac{\boldsymbol{P}\cdot(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^{3}}\,d\tau \tag{12}$$

where $k_e = 1/(4\pi\varepsilon_0)$.

Let us now assume a double layer of thickness h and area S. Taking a volume in the transition zone in the form of a pillbox (small straight cylinder of height h and cross-section dS), one has that $d\tau = h\,dS$ and $\boldsymbol{P} = d\boldsymbol{p}/d\tau = d\boldsymbol{p}/h\,dS$. If $d\tau \to 0$, with $h \to 0$ and defining $\boldsymbol{P}_s = \lim_{h\to 0}\boldsymbol{P}\,h = d\,\boldsymbol{p}/dS$ as the dipole moment per unit surface, Equation (12) takes the expression:

$$V(r) = k_e \int_{S} \frac{\boldsymbol{P}_s\cdot(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^{3}}\,dS = k_e \int_{S} P_s\,d\Omega \tag{13}$$

where it is assumed that $\boldsymbol{P}_s$ and $d\,\boldsymbol{S}$ are collinear.

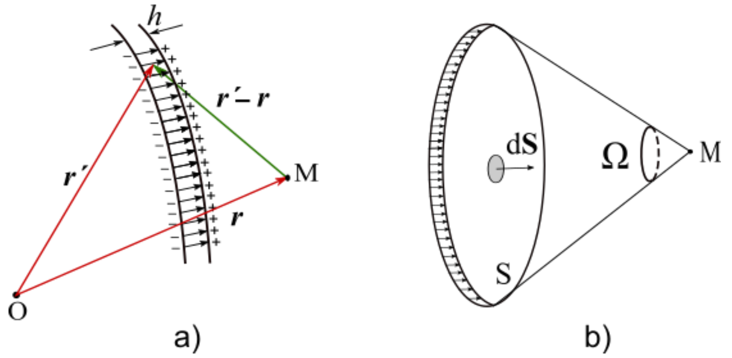

Fig. 2. a) Dielectric double layer (general case); b) Dielectric double layer when polarization surface density is a constant.

If the polarization surface density is assumed constant,

$$V(r) = k_e\ P_S\ \Omega\,(\boldsymbol{r}) \tag{14}$$

where $\Omega\,(\boldsymbol{r})$ is the solid angle at which the surface S (of the wave-front) is viewed from M. In other words, the double layer is a discontinuity surface of the potential, so that if V^{+} is the potential of the positive side and V^{-} of the negative side of the layer, it is satisfied that $V^{+} - V^{-} = 4\pi\, k_e\, P_S = P_S\,/\,\varepsilon_0$ (See Fig. 2.b). In the general case, the dielectric double layer seems to be an appropriate wave-front model when the medium exhibits dielectric anisotropy and magnetic isotropy (See Fig.3).

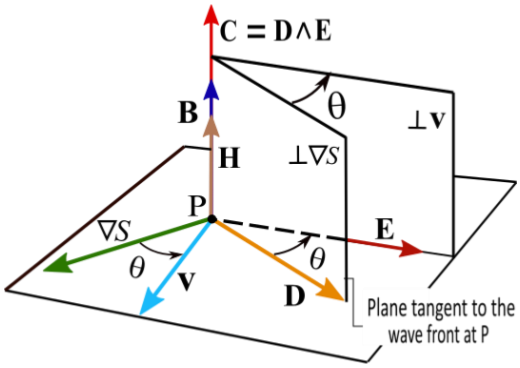

Fig. 3. Electromagnetic fields and ***C*** vector in media with dielectric anisotropy and magnetic isotropy.

2.c. "Thin Plate" model analogy

Since we are mainly concerned with anisotropic media, the so called "soap film" interface model [13], consisting of one single completely flexible layer, does not seem appropriate, as it does not account for all the inherent properties of these media. For instance, the wave-front would carry linear momentum and energy, but no torque. In brief, the vanishing thickness $(h \to 0)$ leads to a vanishing ***C***.

Then, it seems reasonable to adopt the double-layer model with a small but finite thickness, that involves a narrow transitional layer [14] widely used in surface phenomena. This model allows us to introduce a surface couple density (couple stress) such that,

$$\boldsymbol{m} = \lim_{h\to 0} \frac{d\boldsymbol{M}_0}{d\tau}\ h = \lim_{h\to 0}\ h\boldsymbol{C} \tag{15}$$

which is equivalent to a couple of force densities, $(\boldsymbol{C}\times\boldsymbol{n})\,/\,2$ and $(\boldsymbol{n}\times\boldsymbol{C})\,/\,2$, responsible for the bending and twisting at each point of the plate (see [9, Fig. 2]).

Since in isotropic media, $\boldsymbol{C} = 0$, only the normal stress causes the wave-front to deform, like surface tension deforms a soap film, leaving this one-layer model restricted to isotropic media [13].

2.d. Vortex Sheet Model Analogy

The main aim of this work is to demonstrate the equivalence (from the electromagnetic point of view) between the two analogue models: the "thin plate" model borrowed from solid mechanics [13] and the "vortex sheet" used in fluid mechanics [15]. The analysis of this model holds for all the others, (by simply changing ***C*** to ***P***), taking a volume in the transition zone in the form of a pillbox with a vanishing thickness (See Fig. 4).

Calculating the torque of the system of forces applied to an elementary volume with respect to a generic point O of the medium (see Fig. 4 and [9, Fig.2]), we have

$$\boldsymbol{M}_0 = \tfrac{1}{2}\int_\tau \boldsymbol{r}\wedge(\nabla\wedge\boldsymbol{C})\ d\tau + \tfrac{1}{2}\int_{\partial\tau} \boldsymbol{r}\wedge(\boldsymbol{C}\wedge\boldsymbol{n})\ dS \tag{16}$$

And since $d\tau = h\,dS$, for $h \to 0 \Rightarrow d\tau \to 0$, integration is reduced to the volume contour $\partial\tau$. Thus,

$$\boldsymbol{M}_0 = \tfrac{1}{2}\int_{S_{lat}} \boldsymbol{r}\wedge(\boldsymbol{C}\wedge\boldsymbol{n})\ dS + \tfrac{1}{2}\int_{S_1} \boldsymbol{r}\wedge(\boldsymbol{C}\wedge\boldsymbol{n}_1)\ dS + \tfrac{1}{2}\int_{S_2} \boldsymbol{r}\wedge(\boldsymbol{C}\wedge\boldsymbol{n}_2)\ dS \tag{17}$$

where S_{lat} denotes the lateral surface, $\boldsymbol{n}_2 = \boldsymbol{n}$ and $\boldsymbol{n}_1 = -\boldsymbol{n}$ .This implies that

$$d\boldsymbol{M}_0 = \frac{1}{2}\,\boldsymbol{r} \wedge \left[(\boldsymbol{C}^+ - \boldsymbol{C}^-) \wedge \boldsymbol{n} \right] d\,S \tag{18}$$

Torque which is equivalent to that of a pair of forces (couple stress).

On the other hand, from Eq. (15) one can write

$$d\boldsymbol{M}_0 = \boldsymbol{C}\,d\tau = (\boldsymbol{C}\,h)\,d\,S = \boldsymbol{m}\,dS \tag{19}$$

Tending $h \to 0$, keeping $\boldsymbol{m}$ constant, $\boldsymbol{C} \to \infty$ (that is to say, within the double layer, $\boldsymbol{C}$ loses its physical meaning and is replaced by $\boldsymbol{m}$), and applying the curl theorem in infinitesimal form:

$$(\nabla \wedge \boldsymbol{C})\,h\,d\,S = \boldsymbol{n}\;dS \wedge (\boldsymbol{C}^+ - \boldsymbol{C}^-) \tag{20}$$

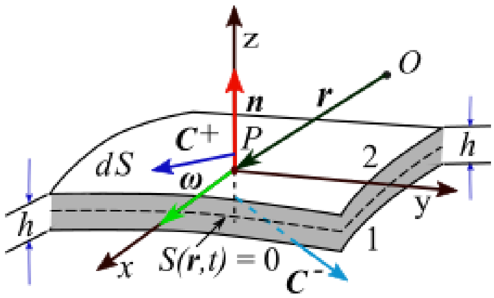

Fig. 4. Gaussian Pillbox of thickness *h* and cross section *d S*, extending just barely on either side of the boundary surface *S*. Nomenclature and parameters for the Vortex Sheet model.

By comparing equations (19) and (20), the analogy arises with a fictitious vortex

$$2\boldsymbol{\omega} = \nabla \wedge \boldsymbol{m} = \boldsymbol{n} \wedge \|\boldsymbol{C}\| \tag{21}$$

Defining, likewise, the infinitesimal divergence as:

$$(\nabla \cdot \boldsymbol{C})\,h = \boldsymbol{n} \cdot (\boldsymbol{C}^+ - \boldsymbol{C}^-) \tag{22}$$

one can write

$$\nabla \cdot \boldsymbol{m} = \boldsymbol{n} \cdot \|\boldsymbol{C}\| \tag{23}$$

Since $\boldsymbol{n} = \nabla S / |\nabla S|$ and taking into account Eq. (6) it is finally obtained that the scalar product $\boldsymbol{n} \cdot \|\boldsymbol{C}\|$ vanishes and $\boldsymbol{C}$ is permanently contained in the plane tangent to the wave-front (see Fig. 5).

To summarize, for the case of a wave-front, one has that $C^+ = 0$, $\boldsymbol{C}^- = \boldsymbol{C}$, so $\|\boldsymbol{C}\| = -\boldsymbol{C}$ and the analogy reduces to the study of a fictitious vector field, $\boldsymbol{m}$, whose sources are given by

$$\left.\begin{aligned} 2\boldsymbol{\omega} = \nabla \wedge \boldsymbol{m} = \boldsymbol{C} \wedge \boldsymbol{n} \\ \nabla \cdot \boldsymbol{m} = 0 \end{aligned}\right\} \tag{24}$$

This model is then analogous to the “vortex sheet” one in fluid dynamics [15], where $\boldsymbol{\omega}$ is the vortex intensity per unit area. It is evident that non-zero values of $\boldsymbol{\omega}$ are associated with a discontinuity of the velocity components normal to $\boldsymbol{n}$. It follows that surfaces across which the tangential velocity changes abruptly are vortex sheets. Figure 5 shows the vortex sheet model of a wave-front.

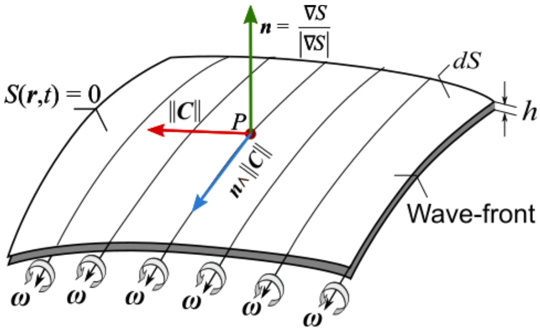

Fig. 5. Vortex sheet model of a wave-front.

3. Conclusions

It is probably not possible to extend, without restrictions, the versatile and successful scalar theory of electromagnetic fields to propagation in non-homogeneous anisotropic media and, consequently, considering the wave-front as a discontinuity surface of the field, it seems necessary to look for models for propagation of interfaces, which allow us to explain -at least qualitatively- the mechanical behavior of wave-fronts in such media.

It is easy to conclude that the single layer models do not solve the problem; in the text, the analogue “soap film” model is ruled out because although it transmits forces, it is incapable of transmitting torques [9].
Forced to use "double layer" models, three of them are presented:

1. "Dielectric double layer", a model useful in anisotropic dielectric media, with magnetic isotropy. We have analyzed only the case in which the surface polarization density is collinear with the normal $\boldsymbol{n}$ to the wave-front.
2. “Thin Plate", a model borrowed from solid mechanics, in the study of the static equilibrium of a thin plate, responsible for the bending and twisting at each point of it. This model satisfies propagation laws of electromagnetic fields, but as it comes from models in statics, is more suitable for stationary processes.
3. "Vortex Sheet": It seems to be a suitable model that includes all the features of wave-front propagation in dielectric/magnetic anisotropic media and that, being a dynamic model from fluid mechanics, shows greater versatility and complexity.

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